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The Roberge-Weiss transition and 't Hooft loops

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Abstract

Roberge and Weiss showed that for $SU(N)$ gauge theories, phase transitions occur in the presence of quark chemical potential which is imaginary. We show that if such a phase transition is of first order, then even with dynamical quarks, 't Hooft loops of arbitrary $Z(N)$ charge are well defined at the phase boundary. We outline the computation of the 't Hooft loop at very high temperature, where semi-classical methods can be used. To leading order in weak coupling, the 't Hooft loop satisfies Casimir scaling in the pure glue theory, but not with quarks. Because the chemical potential is imaginary, typically the interaction measure is negative on one side of the phase transition. Using a matrix model to model the deconfining phase transition, we compute the properties of the Roberge-Weiss phase transition for heavy quarks.

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I. INTRODUCTION

Understanding the nature of the phase diagram of Quantum Chromodynamics (QCD) is one of the outstanding problems in nuclear physics. At zero chemical potential, the theory can be studied by numerical simulations on the lattice. Generally this is not possible at nonzero chemical potential because of the sign problem [1].

One way to trying to understand the theory for nonzero chemical potential is to consider a chemical potential which is purely imaginary: then there is no sign problem, and numerical simulations are possible. Although an imaginary chemical potential is not directly physical, results can be related to those for real chemical potential using Fourier and Laplace transformations [2].

Roberge and Weiss first considered imaginary chemical potential, ϕ, and showed that the $Z(N)$ dynamics of a $SU(N)$ gauge theory in an essential way [2]. They showed that at high temperature, there are first order transitions in the ϕ plane. As the temperature is lowered, these transitions can be of second order or crossover, depending upon the detailed nature of deconfinement with dynamical quarks. This has been studied in effective theories [3], in holographic models [4, 5], and with numerical simulations on the lattice [6, 7].

In this paper we show that Roberge-Weiss phase transitions have an unexpected connection with pure glue theories. Consider $SU(N)$ gauge theories without dynamical quarks. Running the Wilson loop in the temporal direction gives the Polyakov loop. This is an order parameter for $Z(N)$ electric charge, whose expectation value vanishes in the confined phase and is nonzero in the deconfined phase. Similarly, the two point function of Polyakov loops exhibits an area law in the confined phase.

The response of the theory to $Z(N)$ magnetic charges is given by the 't Hooft loop [8–12]. Its behavior is converse to that of the Wilson (Polyakov) loops, which obeys an area law only in the deconfined phase. The string tension for the area law of the 't Hooft loop [9, 10] is equal to the order-order interface tension for $Z(N)$ interfaces [13, 14]. (Throughout this paper we assume that the 't Hooft loop is purely spatial; with or without quarks, temporal 't Hooft loops do not develop an area law at any temperature [11].)

Dynamical quarks carry $Z(N)$ electric charge, and so modify the behavior of both loops. The Polyakov loop is no longer a strict order parameter, but is nonzero at any temperature $T \neq 0$. Similarly, one expects that the 't Hooft loop acts as in the deconfined phase of the

pure glue theory, and so exhibits an area law. However, it is not clear how to define the 't Hooft loop in a theory with dynamical quarks [9, 10].

In this paper we show that at a Roberge-Weiss phase transition that 't Hooft loops can be precisely defined for arbitrary $Z(N)$ charge. We stress that 't Hooft loops are only well defined at the point where Roberge-Weiss transition(s) occur. Nevertheless, we find it of interest that at least at isolated points, that it is possible to do so in a gauge theory with dynamical quarks.

Establishing this result is not difficult. In a gauge theory, a global $Z(N)$ transformation is an overall rotation of the quark field by a constant phase. This can be exactly compensated by a shift in an imaginary chemical potential for the quarks. What is less obvious is how the boundary conditions of the Roberge-Weiss transition are precisely equivalent to those for a $Z(N)$ interface, and thus, to the 't Hooft loop [9, 10].

We compute the behavior of the 't Hooft loop at asymptotically high temperature, where the calculation can easily be done using semi-classical techniques. In the pure glue theory, at leading order and up to corrections $\sim g^3$, one finds that the interface tension associated with 't Hooft loops of different charges satisfies Casimir scaling [15, 16]. We find that even to leading order, dynamical quarks do not respect the Casimir scaling found in the pure glue theory.

We also consider the thermodynamics of Roberge-Weiss phase transitions. We find that the interaction measure is negative on at least one side of the Roberge-Weiss transition. This happens because both the chemical potential and the associated quark number densities are imaginary. Thus their contribution to the energy density can be negative, and so unphysical.

It is also of interest to know how the 't Hooft loop behaves for temperatures which are not asymptotic. To study this, we consider heavy quarks, where the deconfining and Roberge-Weiss phase transitions remain of first order. We use a matrix model [18**?** , 19], which was previously used to study the transition for heavy quarks [20]. We study the theory in the limit where the deconfining transition remains of first order with in the absence of an imaginary quark chemical potential.

In Sec. (II) we discuss the global $Z(N)$ symmetries of the pure gauge theory and how such symmetries arise with dynamical quarks. We also discuss the relationship to 't Hooft loops. In Sec. (III B) consider the thermodynamics of Roberge-Weiss phase transitions.

II. $Z(N)$ SYMMETRY AND 'T HOOFT LOOPS

A. General analysis

We work in imaginary time τ at a temperature T, so $\tau : 0 \to 1/T$. Under a gauge transformation $U(\vec{x},\tau)$, the gluons and quarks transform as

$$A_\mu(\vec{x},\tau) \to \frac{1}{ig}\, U(\vec{x},\tau)\, D_\mu\, U^\dagger(\vec{x},\tau) \quad , \quad q(\vec{x},\tau) \to U(\vec{x},\tau)\, q(\vec{x},\tau) \ , \tag{1}$$

$D_\mu = \partial_\mu + igA_\mu$.

A special class of gauge transformations are those which are symmetric up to elements of $Z(N)$. Consider the matrix $\mathrm{e}^{2\pi ik/N}\mathbf{1}_N$; this has determinant one, and so is an element of $SU(N)$. Since it is proportional to the unity matrix, we can consider aperiodic gauge transitions

$$U(\vec{x},1/T) = \mathrm{e}^{2\pi ik/N} U(\vec{x},0) \ , \tag{2}$$

where k is an integer. Gluons must be periodic in τ, but this aperiodicity in the U's does alter the gluon boundary conditions, cancelling between U and $U^\dagger$.

Quarks are fermions, and so in the absence of a chemical potential, are anti-periodic in τ. The boundary conditions are altered by the presence of a chemical potential, μ. Consider the analytic continuation from a real chemical potential, μ, to one which is imaginary and proportional to the temperature,

$$\mu = 2\pi i\, \phi\, T \ . \tag{3}$$

The quarks now satisfy

$$q(\vec{x},1/T) = -\,\mathrm{e}^{2\pi i\,\phi}\, q(\vec{x},0) \ . \tag{4}$$

Because of the change in the fermion boundary conditions, when $\phi \neq 0$ the theory can exhibit unphysical behavior. For example, $\phi = \frac{1}{2}$ turns fermions into bosons. We shall see an example of this unphysical behavior later, when we find that the interaction measure is negative on one side of the Roberge-Weiss transition line.

Since the $Z(N)$ transformation of Eq. (2), is proportional to the unit matrix, for the quarks it just alters them by an overall phase. Consequently, when $\phi \neq 0$, a $Z(N)$ transformation can be absorbed by a shift in ϕ,

$$\phi \to \phi + \frac{k}{N} \ . \tag{5}$$

The action is invariant under this transformation. By construction, the boundary conditions are as well. Thus it is manifestly a symmetry of the theory, and in particular of the partition function,

$$Z(\phi) = Z\left(\phi + \frac{k}{N}\right) \ , \tag{6}$$

This is the Roberge-Weiss symmetry [2].

This periodicity can also be understood topologically. In the pure glue theory, the gauge group is $SU(N)/Z(N)$, and so there is a global symmetry of $Z(N)$. When $\phi = 0$, the gauge group is just $SU(N)$, and this global $Z(N)$ symmetry is lost. However, by introducing an imaginary chemical potential, ϕ, for free quarks we gain an extra global symmetry of $U(1)$. The coupling of gauge fields reduces this $U(1)$ symmetry to one of $Z(N)$.

We thus have global $Z(N)$ symmetries both in the pure glue theory and, when $\phi \neq 0$, for dynamical quarks. Thus we might expect that there are non-contractible loops which measure windings in these global $Z(N)$ symmetries. In the pure glue theory, such windings are measured by the 't Hooft loop. Since the transformation of ϕ is so intimately tied to the $Z(N)$ symmetry of the pure glue theory, it is perhaps not so surprising that the 't Hooft loop continues to measure such windings, even in the presence of dynamical quarks. Where in the phase diagram is less obvious, and is addressed next.

B. Semi-classical analysis

In this subsection we work in the limit of high temperature, where all computations can be done semiclassically. Many of our conclusions are more general, but it is useful having explicit expressions to understand the relevant physics.

We first need to understand how to generate arbitrary $Z(N)$ transformations from the $SU(N)$ algebra. To do so, consider the diagonal matrix

$$\mathbf{Y}_k = \frac{1}{N}\,\mathrm{diag}\left(k \ldots k, -N+k, \ldots -N+k\right) \ . \tag{7}$$

There are $N-k$ elements with entry k/N, and k elements equal to $-1+k/N$. Thus Y_k has zero trace and is an element of the Lie algebra of $SU(N)$. It is analogous to hypercharge matrix for three flavors. Since

$$\mathrm{e}^{2\pi i \mathbf{Y}_k} = \mathrm{e}^{2\pi i k/N}\,\mathbf{1}_N \ , \tag{8}$$

$\mathbf{Y}_k$ generates a $Z(N)$ transformation with strength k.

The thermal Wilson line is given by

$$\mathbf{L}(\vec{x}) = \exp\left(ig \int_0^{1/T} A_0(\vec{x},\tau) d\tau \right) \ . \tag{9}$$

Then the path

$$A_0 = \frac{2\pi T}{g}\, q\, \mathbf{Y}_k \ , \tag{10}$$

takes one from the ordinary perturbative vacuum for $q = 0$, $\mathbf{L} = \mathbf{1}_N$, to the kth $Z(N)$ transform thereof, for $q = 1$, $\mathbf{L} = \mathrm{e}^{2\pi i k/N}\mathbf{1}_N$.

At the outset it is also useful to consider the matrix $\mathbf{Y}_{N-k}$, where

$$\mathbf{Y}_{N-k} = \frac{1}{N}\mathrm{diag}\left(N-k \ldots N-k, -k \ldots -k\right) \ ; \tag{11}$$

this has k elements with entry $1 - k/N$, and $N - k$ elements with value $-k/N$. By simply reordering the elements, though, it is clear that

$$\mathbf{Y}_{N-k} = -\ \mathbf{Y}_k \ , \tag{12}$$

and so

$$\mathrm{e}^{2\pi i \mathbf{Y}_{N-k}} = \mathrm{e}^{2\pi i(-k)/N}\ \mathbf{1}_N \ . \tag{13}$$

That is, $\mathbf{Y}_{N-k}$ generates a $Z(N)$ transformation with charge $N - k$, which by the additive $Z(N)$ symmetry is equivalent to charge $-k$. This symmetry is useful, because we expect that the interface tension for charge k should be equal to that for charge $N-k$. The matrix $\mathbf{Y}_{N-k}$ makes this manifest, although our final expressions do not obviously reflect this symmetry.

Classically there is no potential for the q's, but one is generated at one loop order, equal to

$$\mathcal{V}_{\mathrm{pt}}^{gl} = \frac{2\pi^2 T^4}{3}\left(-\ \frac{N^2-1}{30} + \sum_{i,j=1}^{N} q_{ij}^2 (1 - |q_{ij}|)^2 \right) \ . \tag{14}$$

The q_i are the elements of $q\mathbf{Y}_k$. The gluon contribution involves the adjoint covariant derivative, and so only the differences of the q_i enter, $q_{ij} = q_i - q_j$.

The q_{ij}'s arise as the gluon energies divided by $2\pi T$. Since the potential for the q_i's arises for a Matsubara sum, then it is periodic in the q_{ij}, $q_{ij} \to q_{ij} + 1$. Consequently, the potential only involves the absolute value of q_{ij}, modulo one. Thus for the gluon term we can always require that $q : 0 \to 1$.

The term independent of q is the just (minus) the pressure of an ideal gas of gluons. For the q_{ij}, there are k elements $= q(k/N)$, and $N - k$ elements $= q(-1 + k/N)$. The potential

arises from the (absolute value) of the $q_i - q_j$'s. For these elements, the $|q_{ij}|$'s are either zero, or q (assuming $q > 0$). There are $2k(N-k)$ such terms which give q, and so the gluon potential is

$$\mathcal{V}^{gl}_{\rm pt}(q) - \mathcal{V}^{gl}_{\rm pt}(0) = \frac{4\pi^2 T^4}{3}\, k(N-k)\, q^2(1-q)^2 \ . \tag{15}$$

The potential is identical at $q = 0$ and $q = 1$, which illustrates the $Z(N)$ symmetry of the pure glue theory between the ordinary vacuum, $k = 0$, and the kth $Z(N)$ transform.

In all, the potential is proportional to $k(N-k)$. This is known as Casimir scaling. It is satisfied in the pure glue theory up to corrections $\sim g^3$ times the leading order term [15, 16].

For a single, massless flavor, quarks contribute to the potential for q as

$$\mathcal{V}^{qk}_{\rm pt} = \frac{2\pi^2 T^4}{3}\left(-\frac{7N}{8} + \sum_{i=1}^{N} q_i^2(1-|q_i|)^2\right) \ . \tag{16}$$

The quark contribution involves the covariant derivative in the fundamental representation, so it is the q_i, and not $q_i - q_j$, which enter. Like the gluon potential, as the quark potential arises from a sum over the Matsubara frequencies, each q_i enters only as the absolute value, modulo one. Thus it is necessary to be careful about the range of the q_i. Even so, clearly Eq. (16) is invariant under two symmetries. As only the absolute value of the q_i enter, one is $q_i \to -q_i$. The second is apparent from the form of the potential: assuming that $1 \geq q_i \geq 0$, the quark potential is also invariant under $q_i \to 1 - q_i$.

For the q_i, there are $N-k$ elements

$$q_i = \frac{1}{2} - \phi + \frac{k}{N}\, q \ ,$$

and k elements

$$q_i = \frac{1}{2} - \phi + \left(-1 + \frac{k}{N}\right) q \ . \tag{17}$$

The q_i are the Euclidean energies divided by $2\pi T$. Thus the first factor of $\frac{1}{2}$ is the πT which arises because quarks are fermions, with boundary conditions which are anti-periodic in imaginary time. The second term, $-\phi$, is the contribution of the imaginary chemical potential. Lastly, the terms $\sim q$ arise from the background field.

While we explicitly compute the potential shortly, most aspects of the physics can be understood without going into such details.

First consider vanishing chemical potential, $\phi = 0$. As discussed above, for the gluons the potential is periodic in q, with $q = 1$ degenerate with $q = 0$, Eq. (15). This reflects the $Z(N)$ symmetry of the pure glue theory.

This is no longer true with dynamical quarks. Then q_i's involve factors of kq/N; for $q = 1$, this is k/N, and generates a nontrivial potential. Computation shows the quark potential has a higher value when $q = 1$ than for $q = 0$. This occurs because quarks in the fundamental representation do not respect the $Z(N)$ global symmetry of the pure glue theory.

Looking just at the q_i's shows how the Roberge-Weiss symmetry of Eq. (6) works. Compare the ordinary perturbative vacuum, where $\phi = q = 0$, to a state where $\phi = k/N$, and $q = 1$. When $\phi = q = 0$, all q_i's equal $\frac{1}{2}$. When $\phi = k/N$, and $q = 1$, all of the q_i's are either $\frac{1}{2}$ or $-\frac{1}{2}$. Since only the absolute value enters, both are equivalent to $\frac{1}{2}$.

Thus we see that a state with $\phi = k/N$ is equivalent to the perturbative vacuum, if we shift the background field by k/N. As we increase ϕ from 0, there is no background field, but at some point, it will shift to that with k/N. The natural point for this to happen is halfway in between, when $\phi = k/(2N)$.

Let us consider the q_i's at this midway point. For $\phi = k/(2N)$ and $q = 0$, all N elements are equal to

$$q_i = \frac{1}{2}\left(1 - \frac{k}{N}\right) \ . \tag{18}$$

When $\phi = k/(2N)$ and $q = 1$, there are $N - k$ elements equal to

$$q_i = \frac{1}{2}\left(1 + \frac{k}{N}\right) \ , \tag{19}$$

and k elements equal to

$$q_i = -\frac{1}{2}\left(1 - \frac{k}{N}\right) \ . \tag{20}$$

Even without computation one can see that the values in Eqs. (19) and (20) are equal to those in in Eq. (18). The q_i's in Eq. (19) are related to those in in Eq. (18) by $q_i \to 1 - q_i$; the q_i's in Eq. (20) are equivalent to those in Eq. (18) under $q_i \to -q_i$.

Explicitly, when $\phi = k/(2N)$, the quark potential is given by

$$\mathcal{V}^{qk}_{\rm pt}(q) = -\frac{4\pi^2}{3}T^4\left((N-k)\left(\frac{1}{4} - \left(\frac{k}{N}\right)^2\left(\frac{1}{2} - q\right)^2\right)\right.$$
$$\left. + k\left(1 - \frac{k}{N}\right)^2\left(\frac{1}{2} - q\right)^2\left(1 - \left(1 - \frac{k}{N}\right)\left|\frac{1}{2} - q\right|\right)^2\right) \ . \tag{21}$$

The values of the potential are clearly equal when $q = 0$ and $q = 1$; note, however, that the second term does involve the absolute value of $\frac{1}{2} - q$. Also, this potential is not simply

proportional to $k(N-k)$, and so does not respect Casimir scaling. As argued above, the potential will respect a transformation under $k \to N-k$, since then $\mathbf{Y}_{-k}$ enters, instead of $\mathbf{Y}_k$.

We can use these results to discuss 't Hooft loops. In the theory without dynamical quarks, Kovner, Korthals-Altes, and Stephanov [9] showed that the 't Hooft loop has a simple physical interpretation. Consider a box which is long in one spatial direction, say that in the z-direction. Put a 't Hooft loop of kth $Z(N)$ charge around the boundary of the box at one end of the box, at $z = L$. Then consider boundary conditions which are $q = 0$ at $z = 0$, and $q = 1$ at $z = L$. These boundary conditions are identical to that for a order-order $Z(N)$ interface of charge k [13–16]. The 't Hooft loop at $z = L$ then forces q to jump from 1 back to 0, so that in all one has periodic boundary conditions. Neglecting this singularity shows that the interface tension for the order-order interface is equal to that for the 't Hooft loop.

Our analysis above shows that across the Roberge-Weiss transition point, for $\phi = k/(2N)$, that one has identically the same boundary conditions. Across this point, the vacuum jumps from $q = 0$ on the left hand side, to $q = 1$ on the right hand side. Consequently, at the Roberge-Weiss transition point, we can define the 't Hooft loop in *precisely* the same manner as in the pure gauge theory.

There are some qualifications which apply. In the pure gauge theory the 't Hooft loop depends only upon the area of the loop, and not upon its shape. With dynamical quarks, the value of the 't Hooft loop depends upon its shape. By construction, we consider a loop which is flat. Because quarks carry electric $Z(N)$ charge, 't Hooft loops with a different shape have different area terms.

We stress that the 't Hooft loop can *only* be defined at the Roberge-Weiss transition point. For example, in the ordinary vacuum, $\phi = 0$, the states with $q = 0$ and $q = 1$ are not degenerate. Typically, the state with $q = 1$ is not even extremal, and there is no way in which to define the 't Hooft loop [9, 10]. Depending upon the matter content, it is possible that the state with $q = 1$ is metastable [21]. In this case, one can define a 't Hooft loop, but the interface tension will have an imaginary part, reflecting this metastability. A necessary condition for the $Z(N)$ loop to be non-contractible is if the $Z(N)$ transformed states are absolutely degenerate, as across a Roberge-Weiss transition.

In the pure gauge theory, the 't Hooft loop measures a non-contractible loop for the

global $Z(N)$ symmetry. As we argued in Sec. (II A), there is also a global $Z(N)$ symmetry if the quarks have an imaginary chemical potential. Since this global $Z(N)$ symmetry of dynamical quarks is intimately tied to the $Z(N)$ transformations of the pure glue theory, it is natural that the 't Hooft loop continues to measure the winding in $Z(N)$.

Our results are valid at asymptotically high temperature, where the Roberge-Weiss phase transition is manifestly of first order. As one lowers the temperature, the Roberge-Weiss transition can become of second order, or even crossover. A single transition line can also split into two, as appears to be indicated by numerical simulations on the lattice [6, 7].

The 't Hooft loop, and the associated winding in $Z(N)$, is well defined at a Roberge-Weiss transition point *if* it is of first order. It is necessary to be at the Roberge-Weiss transition point so that $Z(N)$ transformed vacua are degenerate. The transition needs to be of first order for the loop to exhibit an area behavior. For a transition of second order, or crossover, the 't Hooft loop does not behave like an area law, but only like the length of the loop.

The explicit computation of the 't Hooft loop follows standard methods. A $Z(N)$ interface has an electric field, which contributes to the action as

$$\frac{4\pi^2 T^2}{g^2 N}\, k(N-k) \int dz \left(\frac{dq}{dz}\right)^2 . \tag{22}$$

The interface tension is determined as a semiclassical tunneling between $q=0$ and $q=1$. In the pure glue theory, this involves the sum of the kinetic term in Eq. (22) and the potential of Eq. (15). Since each term is proportional to $k(N-k)$, the interface tension is as well, and so respect Casimir scaling, at least to leading order in the coupling constant.

With dynamical quarks, the kinetic term remains as in Eq. (22), but now the potential is a sum of the gluon term in Eq. (15) and the quark term in Eq. (21). Since the quark potential is not proportional to $k(N-k)$, the interface tension will be a more complicated function of k.

The appearance of Casimir scaling is to some extent an observation about the structure of the theory in weak coupling. Computation to corrections $\sim g^3$ beyond that of leading order shows a small violation of Casimir scaling even in the pure glue theory [15, 16]. So there is nothing fundamental in Casimir scaling, nor in that it is violated by dynamical quarks. The only symmetry principle which must be respected is that for $Z(N)$ periodicity. This requires that the interface tension for k is the same for $N-k$; this is equivalent to $-k$, which as we have argued, is valid.

C. Minimal path for the 't Hooft loop

Before discussing how to compute the 't Hooft loop with dynamical quarks, we have to be careful in considering the paths which are possible. We have assumed that the path for the interface tension is along the direction $\mathbf{Y}_k$, but in principle the path could move in other directions. For an $SU(N)$ gauge theory, the general path lies in the subspace of all commuting generators; this is the Cartan subalgebra, which has $N-1$ dimensions.

For the pure gauge theory one can show that the straight line path is minimal [15, 16]. This has deep geometric reasons: the $\mathbf{Y}_k$ form the boundary of the Weyl chamber, which is the smallest possible region to describe the Cartan subalgebra [18]. The endpoints of $\mathbf{Y}_k$ are the relevant endpoints for the interface tension, so then it is natural that the boundary, along $\mathbf{Y}_k$, is the minimal path which connects these two points.

With dynamical quarks, the structure of the Weyl chamber at the Roberge-Weiss transition is more involved. We have not been able to answer this question for an arbitrary numbers of colors, and so satisfy ourselves with working out the simplest possible cases, working up from $N=2$ to $N=4$.

The case of two colors is trivial. There is only one direction, along $\mathbf{Y}_1 \sim \sigma_3 \sim \mathrm{diag}(1,-1)$, and so the path necessarily lies along $\mathbf{Y}_1$.

The first nontrivial case arises for three colors, where there are two directions in the Cartan sub-algebra. For the quarks, the q_i's are

$$q_i = \left(\frac{1}{2} - \phi\right)\mathbf{1}_3 + \frac{q_3}{2}\,\mathrm{diag}(1,-1,0) + \frac{q_8}{3}\,\mathrm{diag}(1,1,-2)\;. \tag{23}$$

In the standard Gell-Mann notation, the directions are $\lambda_3 \sim (1,-1,0)$ and $\lambda_8 \sim \mathbf{Y}_1 \sim (1,1,-2)$, with associated coordinates q_3 and q_8. For three colors, there is only one interface tension, as $k=-2$ is equivalent to $k=1$. The endpoints of the interface are given by $q_8 = 0$ and 1, with $q_3 = 0$. The straight line path is along $\mathbf{Y}_1$, with one transverse direction, along λ_3.

The q_i's of Eq. (23) are for quarks, but we can use them for gluons, since only the differences of the q_i's enter in the gluon potential, through $q_{ij} = q_i - q_j$. Numerically we find that for the gluon potential of Eq. (14), the straight line path along $\mathbf{Y}_1$ is minimal. That is, for the path where $q_8 \neq 0$ and $q_3 = 0$, for every value of q_8 the potential is minimal with respect to variations in the transverse direction, along q_3. As noted, this is because $\mathbf{Y}_1$ is the boundary of the Weyl chamber for three colors.

Now consider the quark potential at the Roberge-Weiss transition point, $\phi = 1/6$, using the q_i's of Eq. (23) in Eq. (16). As in the pure glue theory for three colors, numerically we have checked that the minimal path is a straight line along $\mathbf{Y}_1$.

For four colors there are two possible interfaces, $k = 1$ and $k = 2$; the associated elements of $Z(4)$ are i and -1, respectively.

For $k = 1$ we can parametrize the q_i's using the usual Cartan generators,

$$q_i = \left(\frac{1}{2} - \phi\right) \mathbf{1}_4 + \frac{q_3}{2}\,\mathrm{diag}(1,-1,0,0) + \frac{q_8}{3}\,\mathrm{diag}(1,1,-2,0) + \frac{q_{15}}{4}\,\mathrm{diag}(1,1,1,-3)\ . \quad (24)$$

The $k = 1$ interface is from $q_{15} : 0$ to 1, with $q_3 = q_8 = 0$ at either end. The straight line path is along $\mathbf{Y}_1 \sim (1,1,1,-3)$, with only q_{15} nonzero. Numerically we checked that the straight line path is minimal, both in the pure glue theory and with dynamical quarks at the Roberge-Weiss point for $k = 1$, where $\phi = 1/8$.

This exercise also shows that the potential has nontrivial structure. At the Roberge-Weiss transition for $k = 1$, $\phi = 1/8$, there is a metastable minimum in the potential when $q_{15} = 0$: it occurs for $q_8 = 1$, with $q_3 = 0$. It is metastable in all three directions, but tunnels with finite lifetime to the usual $Z(4)$ vacua, which are absolutely stable. Such metastable vacua are known to arise for these types of potentials [21].

Lastly we consider four colors with $k = 2$. For the q_i's we take

$$q_i = \left(\frac{1}{2} - \phi\right) \mathbf{1}_4 + \frac{q_2}{2}\,\mathrm{diag}(1,-1,0,0) + \frac{q_2'}{2}\,\mathrm{diag}(0,0,1,-1) + \frac{q_4}{4}\,\mathrm{diag}(1,1,-1,-1)\ . \quad (25)$$

For the interface with $k = 2$, $q_4 = 0$ at one end and $q_4 = 1$ at the other, with $q_2 = q_2' = 0$ at both ends. The straight line path is along $\mathbf{Y}_2 \sim (1,1,-1,-1)$.

To determine stability of a path it is essential to have a parameterization in three independent directions. Our path is along $\mathbf{Y}_2 \sim (1,1,-1,-1)$. The diagonal matrix $\sim (1,-1,0,0)$ is a generator for (the diagonal part of) $SU(2)$ in the first two colors, and is obviously transverse to $\mathbf{Y}_2$. To determine the remaining direction, one can use brute force: one computes the linear combination of $(1,1,-2,0)$ and $(1,1,1,-3)$ which is transverse to $(1,-1,0,0)$ and $(1,1,-1,-1)$. The answer, as in Eq. (25), is $\sim (0,0,1,-1)$. This is just the $SU(2)$ type generator for the third and fourth and fourth colors of $SU(4)$. At least after the fact, this is obvious.

By explicit computation, again one finds that the minimal path is a straight line, with $q_4 \neq 0$ and $q_2 = q_2' = 0$. This is true both for the pure glue theory, and for the theory with

(massless) dynmical quarks at the Roberge-Weiss transition point, $\phi = 1/4$. We did not find metastable minima when $k = 2$.

The examples of $k = 1$ for three colors, and $k = 1$ and $k = 2$ for four colors, suggests that for $SU(N)$ at the Roberge-Weiss transition point(s), $\phi = k/(2N)$, that the path for the associated interface tension is *always* a straight line along $\mathbf{Y}_k$. At present, we can only suggest this as a conjecture and have no general proof. If true, surely it is due to the nature of the Weyl chamber with dynamical quarks at the Roberge-Weiss transition point(s).

If the conjecture of a straight line path is true, it is then immediate to compute the 't Hooft loop. As for the pure glue theory, one has a path in one dimension, along $\mathbf{Y}_k$. One is left with a tunneling problem in this one dimension, and it is easy to solve this tunneling problem by using "energy" conservation for the associated problem in quantum mechanics [14]. This involves both the potential and the kinetic term for the gluons, Eq. (22). For the pure glue theory, as both terms are $\sim k(N-k)$, and the interface tension follows immediately. With dynamical quarks, since the quark contribution to the potential is more involved, even with the conservation of energy one is left with a single integral over q which needs to be computed numerically. Certainly the interface tension with dynamical quarks does not satisfy Casimir scaling.

III. THERMODYNAMICS OF ROBERGE-WEISS TRANSITIONS

A. High temperature

The total pressure p and the entropy density s are a sum,

$$p = p_g + p_q, \qquad s = s_g + s_q, \tag{26}$$

where p_g and s_g are the gluon contributions, and p_q and s_q the quark contributions.

On the left side of the Roberge-Weiss transition, $\phi = k/(2N)$, the gluon contribution is

$$p_g = (N_c^2 - 1)\frac{\pi^2 T^4}{45} \;\; ; \;\; s_g = 4\,\frac{p_g}{T}, \tag{27}$$

while the quarks contribute

$$
\begin{aligned}
p_q &= \frac{\pi^2 N N_{\rm f} T^4}{3}\Big[\frac{7}{60} - 2\Big(\frac{k}{N}q-\phi\Big)^2 + 4\Big(\frac{k}{N}q-\phi\Big)^4\Big] \ ; \\
s_q &= 4\,\frac{p_q}{T} \ ; \\
{\rm Im}(n_q) &= -\frac{2\pi N N_{\rm f} T^3}{3}\Big[\Big(\frac{k}{N}q-\phi\Big) - 4\Big(\frac{k}{N}q-\phi\Big)^3\Big],
\end{aligned}
\tag{28}
$$

Here $kq/N-\phi$ is defined between $-1/2$ and $1/2$. The quark number density is imaginary because the chemical potential is.

To define the energy density, we take the standard expression for a real chemical potential, and assume it remains valid for an imaginary chemical potential:

$$
e = -p + sT + \mu n_q = -p + sT - 2\pi T\ \phi\ {\rm Im}(n_q). \tag{29}
$$

Now consider how thermodynamic functions change on either side of the Roberge-Weiss transition, $\phi = k/(2N)$. On the left side of the transition, $q = 0$, while on the right, $q = 1$, so $kq/N - \phi = \mp k/(2N)$ changes sign across the transition. The pressure and the entropy density are even in $kq/N-\phi$ and do not change. The quark number density is odd in $kq/N-\phi$ and so changes sign. This change in sign for the (imaginary) quark number density is the only reason why the energy density changes at the Roberge-Weiss point and makes the transition of first order.

If we assume that ϕ and T are both fixed, at asymptotically high temperature the contribute of the quark number density to the internal energy density is:

$$
\frac{e}{T^4} \to 3\frac{p}{T^4} - 2\pi\phi\frac{{\rm Im}(n_q)}{T^3} \tag{30}
$$

Thus at high T the interaction measure is due entirely to the contribution from the quark number density,

$$
\Delta \equiv \frac{e-3p}{T^4} = -2\pi\phi\frac{{\rm Im}(n_q)}{T^3}. \tag{31}
$$

The interaction measure is nonzero even at high T because we assume that the (imaginary) chemical potential is proportional to temperature, $\mu = 2\pi i T\phi$. As discussed, it also flips sign across the transition. This holds not only at high T, but persists down to temperatures close to the transition temperature, as we see in the model calculations which follow.

B. Non-perturbative models

The perturbative potential for the q's is given by Eq. (14). It involves the function $V_2(x) = x^2(1-|x|)^2$, and $q_{ij} = (q_i - q_j)_{\rm mod\ 1}$.

The minimum of this potential is always the usual perturbative vacuum, or a $Z(N)$ transform thereof. To model the transition to deconfinement, we can add, by hand, non-perturbative terms [18**?** –20]. The involves one new function, $V_1(x) = |x|(1-|x|)$, where like $V_2(x)$, this function is defined to be periodic in x, modulo one.

For all values of N, a successful fit was obtained with the form

$$\mathcal{V}_g^{np} = -\frac{4\pi^2}{3}\,T^2\,T_d^2 \sum_{i,j}\left(-\frac{c_1}{5}\,V_1(q_{ij}) - c_2\,V_2(q_{ij}) + \frac{N^2-1}{60}\,c_3\right) \tag{32}$$

The parameters c_1 and c_2 are assumed to be independent of temperature, while the temperature dependence of c_3 is simply

$$c_3(T) = c_3(\infty) + (c_3(T_d) - c_3(\infty))\left(\frac{T_d}{T}\right)^2 . \tag{33}$$

Driven by the lattice data, most terms in the non-perturbative potential are $\sim T^2$. The temperature dependence of c_3 also incorporates a MIT bag constant $\sim c_3(T_d) - c_3(\infty)$.

The parameters of the model are fit by the transition in the pure glue theory, where we assume $T_d = 270$ MeV. At the outset, the model involves four parameters, c_1, c_2, $c_3(T_d)$ and $c_3(\infty)$. One then requires two conditions. First, that the transition occurs at T_c. Second, that the pressure vanishes at T_d. The first is a reasonable condition at any N. The second is an approximation, modeling that the pressure is suppressed by $1/N^2$ in the confined phase, relative to that in the deconfined phase. This leaves two parameters, which are fit by the value of the latent heat, and the fall off of the pressure with temperature at asymptotically high T.

To include quarks, we follow Ref. [20] and simply add the one loop term for quarks in the background field. This is then the only way in which the (imaginary) chemical potential enters. For constant quark mass, this contribution is

$$\mathcal{V}_q = 2T{\rm tr}_{f,c}\int\frac{dp^3}{(2\pi^3)}\Big[\ln\Big(1+e^{-\beta\{E_f - i2\pi T(\phi+q_c)\}}\Big) + \ln\Big(1+e^{-\beta\{E_f + i2\pi T(\phi+q_c)\}}\Big)\Big], \tag{34}$$

where ${\rm tr_{f,c}}$ represents the trace over flavor and color and $E_f(p) = \sqrt{p^2+m_f^2}$. We assume that the up and down quarks are isospin symmetric. We thus adopt the notation that the light (up and the down) quark masses are m_l, and the strange quark mass is m_s.

For light quark masses there is a back from to the gluon potential for chiral symmetry breaking or restoration, but we can neglect this here. Such back reaction may be discussed by using the functional renormalization group or the gluon and ghost potentials from the Landau gauge gluon and ghost propagators [22]. When we consider the small quark mass region, we should also consider the meson and also the baryon contributions, but inclusion of these effects is much more involved, and will be treated later.

We shall work in the limit of heavy quark masses, where the light and strange quark masses are constant, and such effects can be neglected. Therefore, as a first step to construct the reliable model of QCD, we investigate the upper part of the Columbia plot at the Roberge-Weiss endpoint.

For later convenience, we introduce the modified Polyakov-loop as

$$\Psi = e^{2\pi i\phi}\,\Phi\ , \qquad \overline{\Psi} = e^{-2\pi i\phi}\,\overline{\Phi}\ . \tag{35}$$

These quantities are useful because they are periodic under the Roberge-Weiss symmetry.

Previously it was found the the pressure and other thermodynamical quantities exhibit unphysical behavior below T_d in the matrix model [20]. This is because the behavior of color singlet quantities, such as glueballs, are not included self-consistently. To handle this, we modify the potential as

$$\mathcal{V}(q) \to \mathcal{V}(q) - \mathcal{V}(q_c) + \mathcal{V}(q_c)\overline{\Psi}\Psi\ , \tag{36}$$

where $q_c = (1/3, -1/3, 0)$. This follows a similar modification in Ref. [20] where the q-dependent part of the potential was modified in one particular direction in color space. The above form is more natural, and suppresses it in a color symmetric manner.

C. Numerical results

For a example for the T-dependence of the imaginary part of the modified Polyakov-loop (Im Ψ), we show the $m_l = m_s$ case. The right(left) panel of Fig. 1 shows the T-dependence of ImΨ for the matrix model for deconfinement (logarithmic Polyakov-loop effective potential).

We can see that there is the tri-critical point in the matrix model for deconfinement, but does not in the logarithmic Polyakov-loop effective potential down to $m_l = m_s = 1$ GeV. There is the possibility that the logarithmic Polyakov-loop effective potential has the

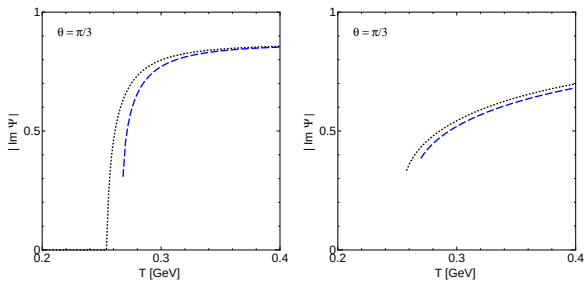

FIG. 1. The T-dependence of Im Ψ. Right (left) panel denotes the result of the matrix model for deconfinement (logarithmic Polyakov-loop effective potential). The dotted and dashed lines are $m_l = m_s = 1$ and 2 GeV, respectively.

tri-critical point in smaller m_l and m_s region, but we should consider the spontaneous chiral symmetry breaking in this case and it is out of our present treatment.

The Columbia plot at $\phi = 1/6$ of the matrix model for deconfinement is shown in Fig. 2.

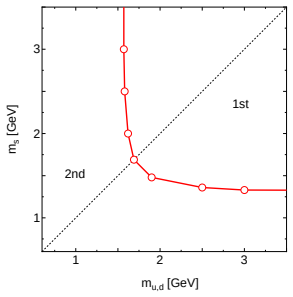

FIG. 2. The Columbia plot at the RW endpoint.

The tri-critical point is determined that the gap of the imaginary and real part of the modified Polyakov-loop vanished at critical T. The first-order region is larger than that at zero chemical potential [20]. It is consistent with recent LQCD prediction. At least in the recent LQCD data [7, 23] suggests that there is the tri-critical point at certain T with two-flavor quarks and thus our matrix model matches this prediction. The critical mass divided by critical temperature (m_c/T_c) are summarized in Table. I. The difference between T_c and T_d is $1-2$ % in the case of $m_l = m_s$. These values are well reproducesd the recent LQCD with the strong coupling and hopping parameter expansions [24]. On the other hand, we can not find the critical mass for the logarithmic Polyakov-loop potential at least down

	$m_l = \infty$	$m_l = m_s$	$m_s = \infty$
Ref. [24]	5.56(3)	6.66(3)	6.25(3)
This model	5.0	6.4	5.9

TABLE I. Summary for m_c/T_c in the recent LQCD simulation with the strong coupling and hopping parameter expansions [24] and the matrix model for deconfinement. We show our result down to first decimal place.

to $m_l = m_s = 1$ GeV. It is the reason why the first-order region of the Columbia plot at $\phi = 1/6$ in Ref. [25] is too wide.

The pressure, internal energy density, entropy density and quark number density with $m_c/T_c = 6.4$ are shown in Fig. 3.

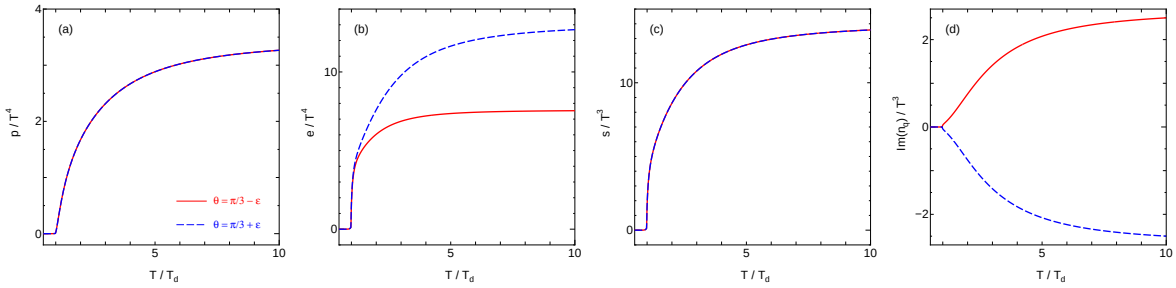

FIG. 3. The T/T_d-dependence of the pressure, energy, entropy and quark number densities with $m_c/T_c = 6.4$ where $T_d = 0.27$ GeV. The solid and dashed lines represents those at $\phi = 1/6 - \epsilon$ and $\phi = 1/6 + \epsilon$, respectively.

To investigate the model dependence, the interaction measure(31) is interesting quantity. The actual result is shown in Fig. 4. We can see that Δ has the two-peak structure at $\phi = 1/6 + \epsilon$. Both lines finally reach the same absolute value with opposite sign as explained in III A.

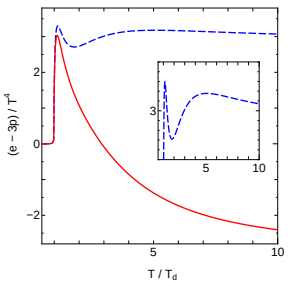

FIG. 4. The T/T_d-dependence of the interaction measure. The definition of the solid and dashed lines is same as Fig. 3.

D. Summary

In this paper we showed that 't Hooft loops of arbitrary $Z(N)$ charge are well defined even with dynamical quarks, at least at the phase boundary of a Roberge-Weiss transition which is of first order. To leading order in weak coupling, the 't Hooft loop satisfies Casimir scaling in the pure glue theory, but not with quarks.

For three colors we computed thermodynamic behavior at large quark mass using an effective matrix model for deconfinement, and considered the form of the Columbia plot. The matrix model well reproduces results obtained by lattice QCD with strong coupling and hopping parameter expansions for m_c/T_c, while models with a logarithmic Polyakov-loop potential do not. We computed the interaction measure about the Roberge-Weiss transition, $\phi = 1/6 \pm \epsilon$, and find that it is enhanced on one side versus the other.

Considering an imaginary chemical potential is clearly useful to discriminate between various effective models of deconfinement. The relationship to the 't Hooft loop suggests that it probes more fundamental aspects of the dynamics in unforeseen ways.

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